Entropy production versus quantum correlations in open quantum systems

Tatiana Mihaescu and <u>Aurelian Isar</u>

Department of Theoretical Physics
National Institute of Physics and Nuclear Engineering
Bucharest-Magurele, Romania
isar@theory.nipne.ro

- The Markovian time evolution of the entropy production (EP) rate is studied as a measure of irreversibility generated in a bipartite quantum system consisting of two coupled bosonic modes immersed in a common thermal environment. The dynamics of the system is described in the framework of the formalism of the theory of open quantum systems based on completely positive quantum dynamical semigroups, for initial two-mode STS, SVS, TS and CS. We show that the rate of the EP of the initial state and nonequilibrium stationary state, and the time evolution of the rate of EP, strongly depend on the parameters of the initial Gaussian state (squeezing parameter and average thermal photon numbers), frequencies of modes, parameters characterising the thermal environment (temperature and dissipation coefficient), and strength of coupling between the modes. We provide a comparison of the behaviour of EP rate and Rényi-2 mutual information in the system.

Entropy production

- Entropy production (EP) is a basic concept in nonequilibrium classical and quantum thermodynamics. It is intimately related to the second law of thermodynamics, which enables identifying and quantifying the irreversibility of physical processes, expressed by the generation of entropy and the dissipation of heat into the surrounding environment of the systems.

- According to the second law of thermodynamics, entropy change ΔS of the state of a system that exchanges energy during its interaction with a thermal environment at temperature T has a lower bound:

$$\Delta S \geq \int \frac{\delta Q}{T}, \qquad (1)$$

where δQ is the infinitesimal heat absorbed by the system. The strict inequality characterises an irreversible process for which energy is dissipated into the environment in the form of heat.

Entropy production

- Besides the entropy that flows from the s. into the reservoir, some additional entropy may be intrinsically generated by the process within the s., called EP. From the second law of thermodynamics, it follows that EP is always non-negative; it only has a zero value when the s. is in thermal equilibrium with its reservoir and it can consequently be used as a measure of the degree of irreversibility of physical processes and to characterise a broad range of nonequilibrium phenomena; it is def. by

$$\Sigma \equiv \Delta S - \int \frac{\delta Q}{T} \geq 0, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = \Pi(t) - \Phi(t), \qquad (2)$$

where $\Pi(t)$ denotes the irreversible EP rate, and $\Phi(t)$ the entropy flux from the s. into the environment. When the s. reaches a stationary state, these two quantities take strictly positive and equal values, while thermal equilibrium is reached only when both are zero.

Master Equation for Two Bosonic Modes Interacting with the Environment

- Markovian Gorini–Kossakowski–Sudarshan–Lindblad master equation for density operator $\rho(t)$:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \frac{1}{2\hbar}\sum_j (2B_j\rho(t)B_j^\dagger - \{\rho(t), B_j^\dagger B_j\}_+). \quad (3)$$

The Hamiltonian of two nonresonant linearly coupled in coordinates bosonic modes

$$H = \frac{\hbar\omega_1}{2}(x^2 + p_x^2) + \frac{\hbar\omega_2}{2}(y^2 + P_y^2) + qxy, \quad (4)$$

where x, y, p_x, p_y are the dimensionless position and momentum operators of the two modes, respectively, and q is the coupling parameter; $\mathbf{R} = \{x, p_x, y, p_y\}^{\mathrm{T}}$, vector of canonically conjugated quadrature operators for the two bosonic modes; σ, 4×4 bimodal covariance matrix with elements :

$$\sigma_{ij} = \mathrm{Tr}[(R_iR_j + R_jR_i)\rho], i, j = 1, \ldots, 4. \quad (5)$$

Covariance matrix

- Lyapunov equation:

$$\frac{d\sigma(t)}{dt} = A\sigma(t) + \sigma(t)A^{\mathrm{T}} + D, \qquad (6)$$

$$A = \begin{pmatrix} -\lambda & \omega_1 & 0 & 0 \\ -\omega_1 & -\lambda & -q & 0 \\ 0 & 0 & -\lambda & \omega_2 \\ -q & 0 & -\omega_2 & -\lambda \end{pmatrix}, \qquad (7)$$

where A denotes the drift matrix, D is the diffusion matrix, and λ is the dissipation rate (we set $\hbar = 1$):

$$D = 2\,\mathrm{diag}\{\lambda\coth\frac{\omega_1}{2k_B T}, \lambda\coth\frac{\omega_1}{2k_B T}, \lambda\coth\frac{\omega_2}{2k_B T}, \lambda\coth\frac{\omega_2}{2k_B T}\}.$$
$$(8)$$

The time-dependent solution of Eq. (6) is

$$\sigma(t) = M(t)[\sigma(0) - \sigma_{\mathrm{s}}]M^{\mathrm{T}}(t) + \sigma_{\mathrm{s}}, \qquad (9)$$

$$M(t) \equiv \exp(At), \quad A\sigma_{\mathrm{s}} + \sigma_{\mathrm{s}}A^{\mathrm{T}} = -D. \qquad (10)$$

$q = 0 \rightarrow$ asympt. Gibbs state (thermal equil.) with env.

Wigner EP rate

- The dynamics of the open quantum systems given by the master equation (3) can be reformulated in terms of the Fokker-Plank equation for Wigner distribution function, therefore it is appropriate to describe the evolution of EP by using a corresponding approach based on the phase space formalism. Consequently, we introduce the Wigner EP rate, given by

$$\Pi(t) \equiv -\partial_t K(W(t)||W_s), \qquad (11)$$

where $K(W(t)||W_s)$ is Wigner relative entropy, $W(t)$ is the time-dependent Wigner function and W_s is Wigner function for the stationary state.

- We introduce the symplectic matrix representing time reversal operator
  $E = \mathrm{diag}(1, -1, 1, -1)$. Then, dynamic variables can be divided according to their time symmetry. Drift matrix A (7) is split into an irreversible component A^{irr}, given by $A^{\mathrm{irr}} = \frac{1}{2}\left(A + EAE^{\mathrm{T}}\right)$, and a reversible one $A^{\mathrm{rev}} = \frac{1}{2}\left(A - EAE^{\mathrm{T}}\right)$:

$$A^{\mathrm{irr}} = \mathrm{diag}\left(-\lambda, -\lambda, -\lambda, -\lambda\right), \qquad (12)$$

$$A^{\mathrm{rev}} = \begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & -q & 0 \\ 0 & 0 & 0 & \omega_2 \\ -q & 0 & -\omega_2 & 0 \end{pmatrix}. \qquad (13)$$

- The analytical expression of EP rate $\Pi(t)$ as a function of drift matrix A, diffusion matrix D, and covariance matrix σ is the following:

$$\Pi(t) = \frac{1}{2}\text{Tr}[\sigma^{-1}(t)D] + 2\text{Tr}[A^{\text{irr}}] + 2\text{Tr}[(A^{\text{irr}})^{\text{T}}D^{-1}A^{\text{irr}}\sigma(t)]. \tag{14}$$

In particular, when the system reaches nonequilibrium stationary state σ_{s}, expression (14) becomes

$$\Pi_{\text{s}} = \text{Tr}[A^{\text{irr}}] + 2\text{Tr}[(A^{\text{irr}})^{\text{T}}D^{-1}A^{\text{irr}}\sigma_{\text{s}}]. \tag{15}$$

Initial state - STS

- We consider an initial squeezed thermal state with covariance matrix

$$\sigma_0 = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ c & 0 & b & 0 \\ 0 & -c & 0 & b \end{pmatrix}, \qquad (16)$$

where

$$\begin{aligned}
a &= 2n_1 \cosh^2 r + 2n_2 \sinh^2 r + \cosh 2r, \\
b &= 2n_1 \sinh^2 r + 2n_2 \cosh^2 r + \cosh 2r, \qquad (17) \\
c &= (n_1 + n_2 + 1) \sinh 2r.
\end{aligned}$$

n_1 and n_2 are the average thermal photon numbers of the modes, and r is the squeezing of the initial state.

Gaussian state ρ of a two-mode continuous variable system can be described by positive Wigner distribution in phase space

$$W_\rho(\boldsymbol{\xi}) = \frac{1}{\pi^2 \sqrt{\det \sigma}} \exp\left(-\boldsymbol{\xi}^\mathsf{T} \sigma^{-1} \boldsymbol{\xi}\right), \tag{18}$$

where $\boldsymbol{\xi} \in \mathbb{R}^4$ and σ is the covariance matrix completely characterising the Gaussian state.

Entropy is usually quantified by using von Neumann entropy. An alternative quantifier of the quantum information contained in a Gaussian state is Shannon entropy of Wigner distribution (18):

$$\mathcal{S}_\sigma(\rho) = \frac{1}{2} \ln(\det \sigma) + 2(1 + \ln \pi). \tag{19}$$

In addition, Rényi-α entropies were introduced in quantum information theory that form a family of additive entropies related to derivatives of the free energy with respect to temperature, defined by:

$$\mathcal{S}_\alpha(\rho) = (1 - \alpha)^{-1} \ln(\operatorname{Tr} \rho^\alpha), \ \alpha \geq 0. \tag{20}$$

Gaussian Rényi-2 Mutual Information (2)

Rényi entropies represent useful instruments for studying quantum correlations in multipartite systems. Up to an additive constant, expression (19) coincides with the Rényi entropy of order 2, given by Eq. (20) for $\alpha = 2$. For $\alpha = 1$ Rényi entropy becomes von Neumann entropy $\mathcal{S}_1(\rho) = -\text{Tr}(\rho \ln \rho)$, and for $\alpha = 2$ from expression (20) we obtain $\mathcal{S}_2(\rho) = -\ln(\text{Tr } \rho^2)$, which is the opposite of the logarithm of purity of the state ρ. Using Eq. (18), we obtain the following expression of the Rényi-2 entropy for Gaussian states:

$$\mathcal{S}_2(\rho) = \frac{1}{2} \ln(\det \sigma). \tag{21}$$

For pure states ($\det \sigma = 1$) $\mathcal{S}_2(\rho) = 0$ and it increases with the mixedness of the state. By comparing the expressions in Eqs (19) and (21), we see that Rényi and Shannon entropy indeed coincide, up to an additional constant. Rényi-2 entropy has all the required properties to be a legitimate measure of entropy, including strong subadditivity.

Gaussian Rényi-2 Mutual Information (3)

For any bipartite Gaussian state ρ of a system with subsystems *A* and *B*, Gaussian Rényi-2 mutual information is defined by

$$\mathcal{I}\left(\rho_{A:B}\right) = \mathcal{S}_2\left(\rho_A\right) + \mathcal{S}_2\left(\rho_B\right) - \mathcal{S}_2\left(\rho\right), \tag{22}$$

where ρ_A and ρ_B are the two marginals of ρ. Two-mode covariance matrix σ is written in block form that contains covariance matrices of the parties:

$$\sigma = \left(\begin{array}{cc} \sigma_A & \sigma_C \\ \sigma_C^{\mathrm{T}} & \sigma_B \end{array} \right), \tag{23}$$

the Gaussian Rényi-2 mutual information has expression

$$\mathcal{I}\left(\rho_{A:B}\right) = \frac{1}{2} \ln \left(\frac{\det \sigma_A \det \sigma_B}{\det \sigma} \right). \tag{24}$$

$\mathcal{I}\left(\rho_{A:B}\right) \geq 0$ and it represents a measure of the total quadrature correlations between the parties *A* and *B* in the state ρ.

The analytical time-dependent expression of Rényi-2 mutual information for the considered system is very complicated; therefore, we only report its expression for the stationary state in the particular case of resonant modes:

$$\mathcal{I}_s = \frac{1}{2}\ln$$

$$\frac{(\lambda^2 + \omega(\omega - q))(\lambda^2 + \omega(\omega + q))(q^4 + 8q^2(\lambda^2 - \omega^2) + 16(\lambda^2 + \omega^2)^2)}{4(q^2(\lambda^2 - \omega^2) + 2(\lambda^2 + \omega^2)^2)^2}.$$
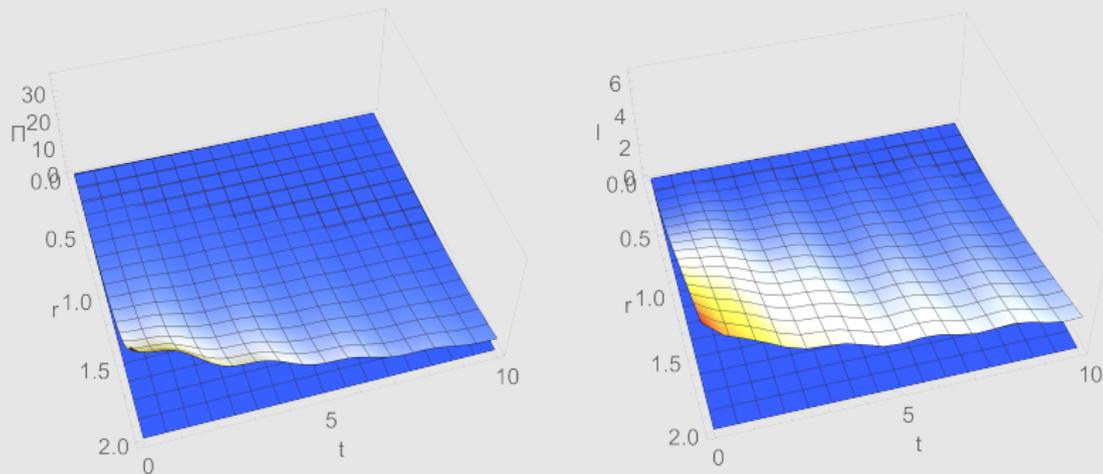
This does not depend on temperature.
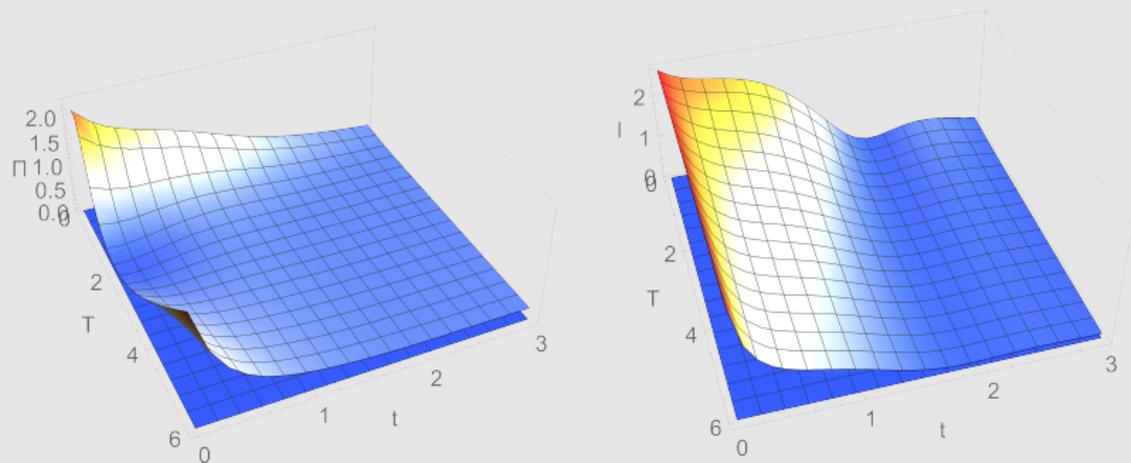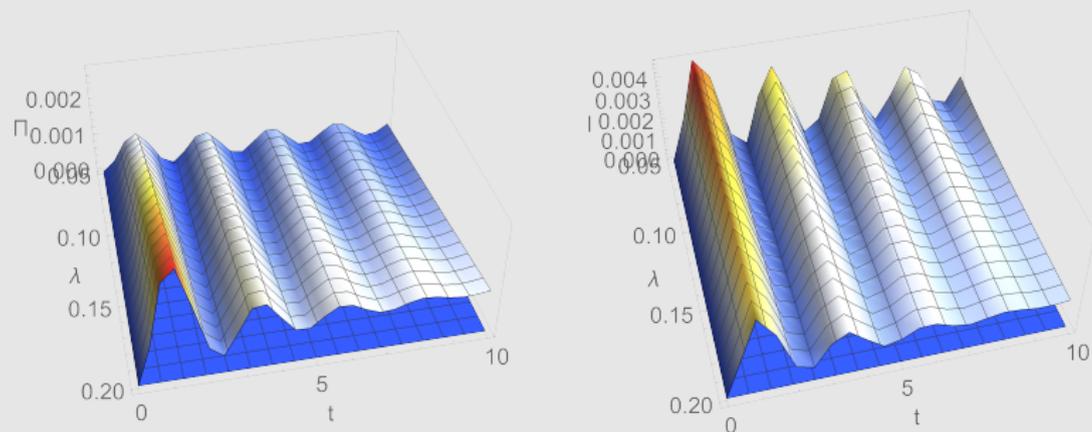
Figure: Π(t) and $\mathcal{I}(t)$

Figure: $\Pi(t)$ and $\mathcal{I}(t)$

Figure: Π(*t*) and 𝓘(*t*)

Figure: Π(t) and 𝓘(t)

Figure: Π and $\mathcal{I}$
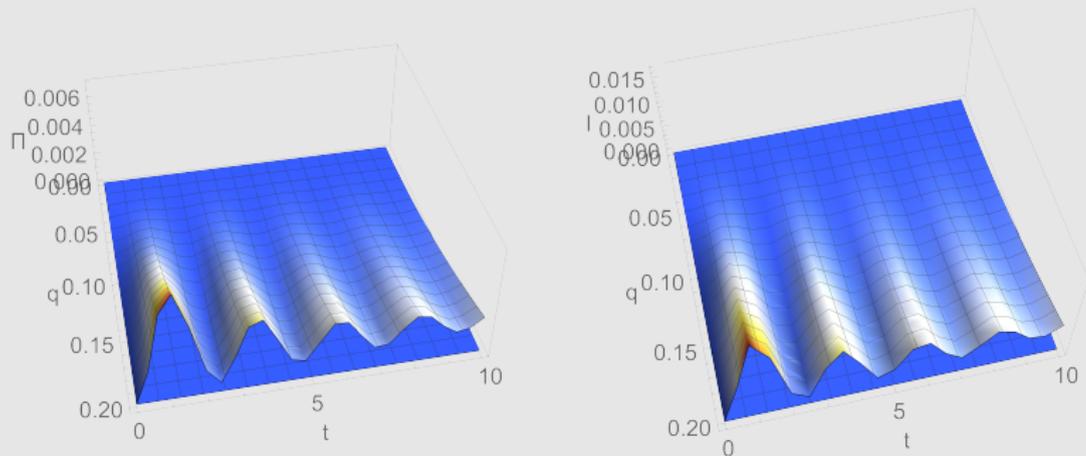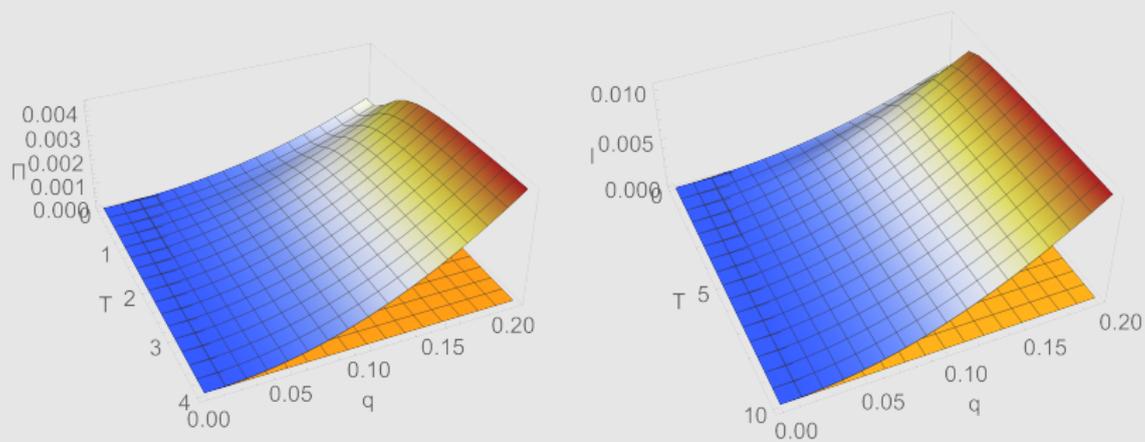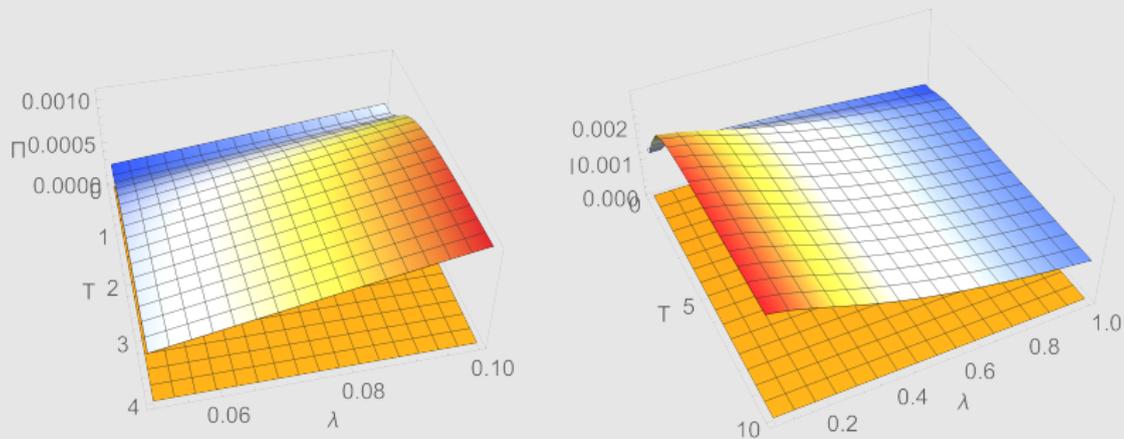
Figure: Π and $\mathcal{I}$

Conclusions

- In order to extend the present analysis, we plan to take into consideration the role played by the squeezing in the thermal reservoir, representing a quantum thermodynamic resource, and perform a similar investigation of the dynamics of entropy production rate in a bipartite system interacting with a squeezed thermal reservoir which manifests additional thermodynamic features compared to the thermal reservoir.

- Obtained results emphasise the closed relation between irreversibility that quantifies the difference from reversible quasistatic transformations generated by the dynamical and stationary process, and correlations existing in the considered bipartite system.

- Tatiana Mihaescu, Aurelian Isar, Entropy, **24**, 696 (2022)

Thank You!