Nonlinear models of Affine Toda Field Theory

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Presentation Overview

Introduction

- A model of scalar field, $\phi(x^\mu)$, in Quantum Field Theory (QFT) can be described through a Lagrangian density of the form:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi).$$

  Depending on the choice of the potential $V(\phi)$, the model can represent many types of physical situations.

- A Toda field theory is a nice example which can appear in mathematics and physics, specifically in the study of field theory and partial differential equations.
- The theory can be specified by a Lie algebra and usually refer to theories with a finite Lie algebra. Depending on the rank of its associated Lie algebra, the Toda field ϕ represents in fact a collection of scalar fields $\{\phi^i, i = 1, ..., r\}$, with r the rank.

Introduction

- In a $2D$-Minkowski spacetime, x, t, the Toda Lagrangian can be written
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{\beta^2} \sum_{i=1}^{r} n_i \exp \beta(\alpha_i \phi^i)$$
The integers n_i are Kac or Dynkin labels, m is the mass and β the coupling constant,
α_i is the i-th simple root, and provides a basis for the Cartan subalgebra on $\mathcal{R}^r$.
- At the classical level, the Toda field theories represent nonlinear integrable models and their solutions describe solitons.
- At the quantum level, by liniarizing the Toda fields we can generate quantum models with the spectrum consisting of one or many massive particles.
- In Cosmology, the Toda fields could describe tachionic fields appearing in inflation models.

Toda fields: Liouville Eq.

- The **Liouville field theory** is associated to a A_1 Cartan matrix, which corresponds to the Lie algebra $su(2)$ with a single simple root.
- Liouville model describes the dynamics of a field ϕ defined by an exponential potential $V(\phi) = \exp(2\beta\phi)$,
- The attached Euler-Lagrange equation is:

$$\phi_{tt} - \phi_{xx} = 2\beta \exp(2\beta\phi)$$

- In Liouville theory, the momentum is not conserved in interactions.

Toda fields: The sinh-Gordon Eq.

- The **sinh-Gordon model** is the affine Toda field theory with the generalized Cartan matrix

$$A_2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

and a positive value for β after we project out a component of ϕ which decouples.

- The sinh-Gordon model described by the Lagrangean:

$$\mathcal{L} = \frac{1}{2}(\phi_t^2 - \phi_x^2) - \frac{2m^2}{\beta^2}\cosh(\beta\phi)$$

- The corresponding Euler-Lagrange equations are:

$$\phi_{xx} - \phi_{tt} = \frac{2m^2}{\beta^2}\sinh\phi$$

Toda fields: The sine-Gordon Eq.

- The **sine-Gordon model** has the same Cartan matrix but an imaginary $\beta = ib$. It corresponds to the Lie algebra $su(2)$. with the potential:

$$V(\phi) = \sum_{i=0}^{1} n_i \exp(\beta \alpha_i \phi) = \exp(\beta\phi) + \exp(-\beta\phi).$$

- The model can be described using as independent variables $\{x, y = it\}$ by:

$$\mathcal{L} = \frac{1}{2}(\phi_x^2 + \phi_y^2) + \frac{2m^2}{b^2}\cos(b\phi).$$

- The corresponding E-L equation will be in this case:

$$\phi_{xx} + \phi_{tt} = \frac{2m^2}{b^2}\sin b\phi$$

It is no longer a soliton equation, but it has many similar properties

Toda fields in QFT

- Toda fields were quantized and describe processes of multiparticle production that cancel on mass shell.
- The **quantum sine-Gordon** equation is somehow similar with its classical version, but contains a parameter that can be identified with the Planck constant. The exponentials become vertex operators:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m\phi^2 - \alpha(V_\beta + V_{-\beta})$$

  with $V_\beta =: e^{i\beta\phi}$:, where the semi-colons denote normal ordering.
- The model is S-dual to the Thirring model (Coleman correspondence) and serves as an example of boson-fermion correspondence in the interacting case.

Toda fields in Cosmology

- Important examples of Toda fields appears in cosmology. The inflation phenomena in the early Universe is obtained when we choose a potential expressed through a **tachyonic field**.
- A tachyonic nonstandard Lagrangian of DBI-type was intensively used by GORAN and the Nis team. It has the potential as a multiplicative factor and a square root of derivatives as a “kinetic” term:

$$\mathcal{L}(\phi, \partial^{\mu}\phi) = -V(\phi)\sqrt{1 + g_{\mu\nu}\partial^{\mu}\phi\partial^{\nu}\phi}.$$

- This model is a particularly attractive for K-inflation, defined by the local action for a scalar field minimally coupled to Einstein gravity, useful, as we said, in describing the very early stage of the Universe.

A generalized Toda potential

A generalized Toda potential that includes Liouville, sinh-Gordon or sine-Gordon models is:

$$V(\phi) = p \exp \phi - \frac{q}{2} \exp(-2\phi). \quad (1)$$

This potential can be generalized as:

$$V(\phi) = p \exp(m\phi) + q \exp(n\phi) \quad (2)$$

It describes:
- In QFT- a quantum field with the spectrum consisting of a single massive particle.
- In the Soliton Theory - a classical model of nonlinear integrable system a zero curvature representation with flat connections, an important feature that assures the existence of soliton-like solutions.

The generalized Toda field equation

Due to the non-polynomial form of (2), special techniques have to be applied in order to get its solutions. The classical approach consists in using the transformation

$$\phi(x,t) = \ln u(x,t). \tag{3}$$

The E-L Eq. corresponding to (2) with the change of variable (3) is the nonlinear NODE:

$$u_{xt}u - u_x v_t + p v^{\sigma+2} + q v^{\rho+2} = 0. \tag{4}$$

The equation (4) contains 4-parameters: the real parameters p, q, respectively the integers σ, ρ. This number can be reduced using suitable choices. We will discuss here a reduced 3-parameters equation which corresponds to the choices $\sigma + 2 = m$, $\rho = -2$. It is known as the generalized DBM equation and has the form:

$$v_{xt}v - v_x v_t + p v^m + q = 0 \tag{5}$$

Direct solving by dimensional reduction

Many approaches have been proposed for solving (5), as the Riemann-Hilbert, the Lax operators or the bilinearization Hirota methods.

A direct solving approach supposes, as a first step, to transform (5) into a nonlinear ordinary differential equation (NODE), using the wave transformation, $\xi = x - Vt$. Now $v(x,t)$ becomes $u(\xi)$ and (5) becomes:

$$-Vuu'' + Vu'^2 + pu^m + q = 0 \tag{6}$$

The attached flow approach

We will go further using here the attached flow method, a method based on three main assumptions:

- the variable $u(\xi)$ must supplementary satisfy a flow equation of the form (classic requirement for reducing the differentiability order recommended by textbooks):

$$u'(\xi) = f(u) \tag{7}$$

- the flow $f(u)$ is connected with the free-derivative term $E(u) \equiv pv^m + q$ via a forced decomposition of the form:

$$E(u) = f(u) \cdot h(u) \tag{8}$$

- the flow is considered as a polynomial and a special balancing request allows to predict its mathematical form.

The generalized Toda-DBM equation with $m = 4$

We will illustrate how the attached flow is working, by considering the equation (6) for the case $m = 4$. The graduation rules leads us to the idea that the flow should have the form:

$$f(u) = \alpha_2 u^2 + \alpha_1 u + \alpha_0 \tag{9}$$

For compatibility with (8), the coefficients α_i satisfy in this case the constraints:

$$V\alpha_2^2 = p, \alpha_1 = 0, \ V\alpha_0^2 = q$$

As an explicite expression of the flow, we can use:

$$f(u) = \pm\sqrt{\frac{p}{V}}u^2 \pm \sqrt{-\frac{q}{V}} \tag{10}$$

Toda solitons for $m = 4$

By using the previous expression for integrating the flow equation (7), we get:

- for $q \neq 0$ and $-\frac{pq}{V^2} > 0$, the solutions are periodic:

$$u_1(\xi) = \left(-\frac{q}{p}\right)^{\frac{1}{4}} + \left(-\frac{q}{p}\right)^{\frac{1}{4}} tg\left(-\frac{pq}{V^2}\right)^{\frac{1}{4}} \xi$$

$$u_2(\xi) = \left(-\frac{q}{p}\right)^{\frac{1}{4}} + \left(-\frac{q}{p}\right)^{\frac{1}{4}} \cot\left[-\left(-\frac{pq}{V^2}\right)^{\frac{1}{4}} \xi\right] \tag{11}$$

- for $q \neq 0$ and $-\frac{pq}{V^2} < 0$ the solutions are hyperbolic:

$$u_3(\xi) = \left(-\frac{q}{p}\right)^{\frac{1}{4}} + \left(-\frac{q}{p}\right)^{\frac{1}{4}} th\left(-\frac{pq}{V^2}\right)^{\frac{1}{4}} \xi$$

$$u_4(\xi) = \left(-\frac{q}{p}\right)^{\frac{1}{4}} + \left(-\frac{q}{p}\right)^{\frac{1}{4}} cth\left[-\left(-\frac{pq}{V^2}\right)^{\frac{1}{4}} \xi\right] \tag{12}$$

- for $q = 0$ the solution is rational:

Conclusions

- The affine Toda field theory - unifies apparently different fields: QFT, Gravity and nonlinear dynamics.
- The key connection point: the Lie algebras that can be attached in all the cases to Toda theories and that assures the existence of soliton-like solutions for the associated Euler-Lagrange equations
- The main focus of this presentation: how to get solitons using the attached flow method.
- The application of the method was illustrated on a specific case of Toda field theory, the case when the Euler-Lagrange equation is similar with the generalized Dodd-Bullough-Mikhailov equation with a nonlinearity of order fourth.

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- **Thank you SEENET for the 20 years of friendship you have given us! Many happy celebrations from now on!**