Thermodynamic length for three-dimensional holographic models and optimal processes

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BWXX SEENET-MTP Meeting,

Vrnjačka Banja, Serbia

August 30, 2023

Today

- Thermodynamic Geometry
- Thermodynamic Length
- Optimal processes in $WAdS_3$

Geometrizing thermodynamics

- The first law of thermodynamics (energy representation):

$$dE = TdS + \sum_{r=2}^{n-1} I_r dE^r = \sum_{a=1}^{n} I_a dE^a = \vec{I}.d\vec{E}. \qquad (1)$$

- The first law of thermodynamics (entropy representation):

$$dS = \frac{1}{T}dE - \sum_{r=2}^{n-1} J_r dS^r = \sum_{a=1}^{n} J_a dS^a = \vec{J}.d\vec{S}. \qquad (2)$$

- Fundamental relations and natural variables:

$$E = E(\vec{E}), \quad S = S(\vec{S}). \qquad (3)$$

- Equations of state:

$$I_a = \frac{\partial E(\vec{E})}{\partial E^a}\bigg|_{\dots,\hat{E}^a,\dots}, \quad J_a = \frac{\partial S(\vec{S})}{\partial S^a}\bigg|_{\dots,\hat{S}^a,\dots} \qquad (4)$$

The space of equilibrium states

The relation $E = (\vec{E})$ defines an n-dimensional surface $\mathcal{E}^n$ embedded in $\mathbb{E}^{n+1}$ with coordinates $\left(E^1, ..., E^n, E\right)$.

$\mathcal{E}^n$ becomes an **equilibrium manifold** if equipped with a proper n-dimensional Riemannian metric.

1. **Hessian thermodynamic metrics**
   F. (Weinhold 1975, G. Ruppeiner 1979)
2. **Legendre invariant thermodynamic metrics**
   (H. Quevedo 2016)
3. **Covariant thermodynamic metrics**
   (L. Velazquez 2012)

Thermodynamic metrics

- G. Ruppeiner 1979 ($\epsilon = \pm 1$):

$$ds^{(R)} = -\epsilon \frac{\partial^2 S(\vec{S})}{\partial S^a \partial S^b} dS^a dS^b. \tag{5}$$

- F. Weinhold 1975:

$$ds^{(W)} = \epsilon \frac{\partial^2 E(\vec{E})}{\partial E^a \partial E^b} dE^a dE^b. \tag{6}$$

- Covariant metric (L. Velazquez 2012):

$$ds^{(V)} = \epsilon (\nabla_a \nabla_b S) dS^a dS^b = \epsilon \left(\frac{\partial^2 S}{\partial S^a \partial S^b} - \Gamma^c_{ab}(g) \frac{\partial S}{\partial S^c} \right) dS^a dS^b. \tag{7}$$

- Legendre invariant metrics (H. Quevedo 2016), $L \in \mathbb{R}$, $k \in \mathbb{Z}$:

$$ds^{(Q,III)} = L \sum_a \left(S^a \frac{\partial S}{\partial S^a} \right)^{2k+1} \left(\frac{\partial^2 S}{\partial S^a \partial S^b} dS^a dS^b \right). \tag{8}$$

Thermodynamic length and irreversibility

- **Thermodynamic length** quantifies the distance between two equilibrium states. It is the number of fluctuations associated with the change of the state of the system. In entropy natural coordinates:

$$\mathcal{L}[\gamma] = \int_\gamma \sqrt{g_{ab}(\vec{S}) dS^a dS^b}. \tag{9}$$

In annfine parametrization with parameter t:

$$\mathcal{L}(\tau) = \int_0^\tau \sqrt{g_{ab}(\vec{S})\dot{S}^a \dot{S}^b}\, dt. \tag{10}$$

- **Thermodynamic divergence** of the path (Cauchy-Schwarz),

$$\mathcal{J} = \tau \int_0^\tau g_{ab}(\vec{S})\dot{S}^a \dot{S}^b dt \geq \mathcal{L}^2, \tag{11}$$

measures the efficiency of the quasi-static protocols.

- **$\mathcal{L}$ sets lower bounds on dissipation!**

Optimal finite-time thermodynamic protocols

Thermodynamic length

$\mathcal{L}$ is a measure of the distance between two macro states on the equilibrium manifold $\mathcal{E}^n$.

Optimal finite-time processes

$\mathcal{L}$ defines optimal quasistatic protocols on $\mathcal{E}^n$.

Efficiency of a process

$\mathcal{L}$ is a measure of the energy required to transform the system from one state to another. It characterizes the geometric path that minimizes the dissipation (or maximizes the efficiency) of a thermodynamic process.

Applications to holographic models

Topological Massive Gravity (TMG)

Topological Massive Gravity (TMG):

$$I_{TMG} = \frac{1}{16\,\pi\,G}\int_{\mathcal{M}} d^3x\,\sqrt{-g}\,\left(R + \frac{2}{L^2}\right) + \frac{1}{\mu}\,I_{CS} + \int_{\partial\mathcal{M}} B. \quad (12)$$

$WAdS_3$ (Annios, Li, Padi, Song, Strominger 2009):

$$ds^2 = \ell^2\big(dt^2 + 2M(r)dtd\theta + N(r)d\theta^2 + D(r)dr^2\big), \quad (13)$$

with the following metric functions:

$$N(r) = M^2(r) - \frac{1}{4D(r)}, \quad (14)$$

$$M(r) = \nu r - \frac{1}{2}\sqrt{r_+ r_-(\nu^2+3)}, \quad (15)$$

$$D(r) = \frac{1}{(\nu^2+3)(r-r_+)(r-r_-)}. \quad (16)$$

Energy representation and dual conformal theory

The first law of thermodynamics:

$$dM = TdS + \Omega dJ, \tag{17}$$

$$M(S,J) = \frac{\sqrt{(5\nu^2+3)\,(3G\nu S^2 + 2\pi^2 J\ell)}}{2\pi\ell\sqrt{3G\nu}} - \frac{\nu S}{\pi\ell}, \tag{18}$$

$$T = \frac{\partial M}{\partial S}\bigg|_J = \frac{S}{2\pi\ell}\left(\sqrt{\frac{3G\nu\,(5\nu^2+3)}{3G\nu S^2 + 2\pi^2 J\ell}} - \frac{2\nu}{S}\right), \tag{19}$$

$$\Omega = \frac{\partial M}{\partial J}\bigg|_S = \frac{\pi}{2}\sqrt{\frac{5\nu^2+3}{3G\nu\,(3G\nu S^2 + 2\pi^2 J\ell)}}. \tag{20}$$

Transfer to the dual conformal theory:

$$G \to \frac{\ell\sqrt{4c_R - 5c_L}}{\sqrt{3c_L}(c_R - c_L)}, \quad \nu \to \sqrt{\frac{3c_L}{4c_R - 5c_L}}. \tag{21}$$

Hessian thermodynamic metrics

- Weinhold ($\epsilon = \pm 1$: elliptic/hyperbolic geometry):

$$dS_W^2 = \epsilon \left(\frac{\partial^2 M}{\partial S^2} dS^2 + 2 \frac{\partial^2 M}{\partial S \partial J} dS dJ + \frac{\partial^2 M}{\partial J^2} dJ^2 \right). \tag{22}$$

Thermodynamic curvature:

$$R_W = \frac{2\sqrt{3}\pi G \nu \ell}{\epsilon \sqrt{G\nu \left(5\nu^2 + 3\right) \left(3G\nu S^2 + 2\pi^2 J \ell\right)}}. \tag{23}$$

- Ruppeiner ($\epsilon = \pm 1$: elliptic/hyperbolic geometry):

$$dS_R^2 = \epsilon \left(\frac{\partial^2 S}{\partial M^2} dM^2 + 2 \frac{\partial^2 S}{\partial M \partial J} dM dJ + \frac{\partial^2 S}{\partial J^2} dJ^2 \right). \tag{24}$$

Thermodynamic curvature:

$$R_R = \frac{\left(\nu^2 + 3\right) \sqrt{3G\nu}}{\pi \epsilon \sqrt{2\ell \left(5\nu^2 + 3\right) \left(6G\nu M^2 \ell - J\left(\nu^2 + 3\right)\right)}}. \tag{25}$$

Weinhold thermodynamic length

In natural parameters:

$$\mathcal{L} = \left(\frac{\pi\epsilon\sqrt{5\nu^2+3}}{2\sqrt{3G\nu}}\right)^{1/2} \int_\gamma \frac{\sqrt{6G\nu dS(JdS - SdJ) - \pi^2 \ell dJ^2}}{\left(3G\nu S^2 + 2\pi^2 J\ell\right)^{3/4}}. \qquad (26)$$

Isentropic processes $S = const$ (not possible for $\epsilon = 1$):

$$\mathcal{L}_S(J_0, J) = \frac{i\sqrt{2\epsilon}\sqrt[4]{(5\nu^2+3)\left(3G\nu S^2 + 2\pi^2 J\ell\right)}}{\sqrt{\pi\ell}\sqrt[4]{3G\nu}}\Bigg|_{J_0}^{J} \in \mathbb{C}. \qquad (27)$$

Also not possible for $\epsilon = -1$: $\mathcal{L}_S < 0$ for $J > J_0$.
OK for $J < J_0$ and $\epsilon = -1$ (hyperbolic information sector).
Processes with $\epsilon > 0$ and constant angular momentum $J = const$:

$$\mathcal{L}_J(S_0, S) = -\frac{S\sqrt{\epsilon}\sqrt[4]{3G\nu(5\nu^3+3)}}{\pi\sqrt[4]{8J\ell^3}}\, {}_2F_1\left(\frac{1}{2}, \frac{3}{4}, \frac{3}{2}, -\frac{3GS^2\nu}{2J\pi^2\ell}\right)\Bigg|_{S_0}^{S} < 0. \qquad (28)$$

Not possible for $S > S_0$, maybe allowed for $S < S_0$?

Geodesic paths on $\mathcal{E}$

Let $x(t) = \big(S(t), J(t)\big)$:

$$\ddot{x}^\sigma(t) + \Gamma^\sigma_{\mu\nu}(g)\dot{x}^\mu(t)\dot{x}^\nu(t) = 0. \tag{29}$$

Geodesic paths do not depend on ϵ (2^{nd} order nonlinear ODEs):

$$\ddot{S} - \frac{\pi^2\ell\dot{J}\dot{S}}{3G\nu S^2 + 2\pi^2\ell J} - \frac{3G\nu S\dot{S}^2}{3G\nu S^2 + 2\pi^2\ell J} = 0, \tag{30}$$

$$\ddot{J} + \frac{3G\nu J\dot{S}^2}{3G\nu S^2 + 2\pi^2\ell J} - \frac{6G\nu S\dot{J}\dot{S}}{3G\nu S^2 + 2\pi^2\ell J} - \frac{3\pi^2\ell\dot{J}^2}{6G\nu S^2 + 4\pi^2\ell J} = 0. \tag{31}$$

Processes with $S = const$ ($J_0 = J(0)$ and $\dot{J}_0 = \dot{J}(0)$):

$$\ddot{J}(t) - \frac{3\pi^2\ell\dot{J}^2(t)}{6G\nu S^2 + 4\pi^2\ell J(t)} = 0 \;\Rightarrow\; J(t) = J_0 + \dot{J}_0 t + J_2 t^2 + J_3 t^3 + J_4 t^4.$$

Processes with $J = const$ are possible only for $J = 0$:

$$\ddot{S}(t) - \frac{\dot{S}(t)^2}{S(t)} = 0 \quad \Rightarrow \quad S(t) = S_0 e^{\frac{\dot{S}_0}{S_0}t}, \quad S_0 = S(0), \quad \dot{S}_0 = \dot{S}(0). \tag{32}$$

Optimal protocols with $S = const$ and $J > J_0$

TD length in time parametrization:

$$\mathcal{L}(\tau) = \int_0^\tau \sqrt{g_{ab}(x)\dot{x}^a\dot{x}^b}\,dt. \tag{33}$$

Let $J(0) = J_0$ and $\dot{J}(0) = \dot{J}_0$, $S = const$ is not allowed for $\epsilon = 1$:

$$\mathcal{L}_S(\tau) = \frac{i\pi^{3/2}\dot{J}_0\sqrt{\ell\epsilon}\sqrt[4]{5\nu^2+3}}{\sqrt[4]{3G\nu}\sqrt{2\left(3G\nu S^2 + 2\pi^2 J_0 \ell\right)^{3/2}}}\tau \in \mathbb{C}. \tag{34}$$

$S = const$ is not allowed for $\epsilon = -1$ due to negative TD length:

$$\mathcal{L}_S(\tau) = -\frac{\pi^{3/2}\dot{J}_0\sqrt{\ell}\sqrt[4]{5\nu^2+3}}{\sqrt[4]{3G\nu}\sqrt{2\left(3G\nu S^2 + 2\pi^2 J_0 \ell\right)^{3/2}}}\tau < 0, \tag{35}$$

since the initial rate of change of the angular momentum $\dot{J}_0 > 0$ for $J > J_0$ (J is increasing with time!)

Optimal protocols with $S = const$ and $J < J_0$

For $\epsilon = -1$ and decreasing momentum $J < J_0$ (**Penrose process**):

$$\mathcal{L}_S(J_0, J) = -\frac{\sqrt{2}\sqrt[4]{(5\nu^2+3)\left(3G\nu S^2 + 2\pi^2 J\ell\right)}}{\sqrt{\pi\ell}\sqrt[4]{3G\nu}}\Bigg|_{J_0}^{J} > 0. \quad (36)$$

In time parametrization with $\dot{J}_0 = -|\dot{J}_0| < 0$:

$$\mathcal{L}_S(\tau) = \frac{\pi^{3/2}|\dot{J}_0|\sqrt{\ell}\sqrt[4]{5\nu^2+3}}{\sqrt[4]{3G\nu}\sqrt{2\left(3G\nu S^2 + 2\pi^2 J_0\ell\right)^{3/2}}}\tau = v\tau > 0. \quad (37)$$

Calculate the deceleration time τ from J_0 to J:

$$\tau = \frac{2X_0^{3/4}\left(\sqrt[4]{X_0} - \sqrt[4]{X}\right)}{\pi^2\ell|\dot{J}_0|}, \quad X|X_0 = 3G\nu S^2 + 2\pi^2\ell J|J_0. \quad (38)$$

The thermodynamic speed v of the process:

$$v = \dot{\mathcal{L}}_S(\tau) = \frac{\pi^{3/2}|\dot{J}_0|\sqrt{\ell}\sqrt[4]{5\nu^2+3}}{\sqrt[4]{3G\nu}\sqrt{2\left(3G\nu S^2 + 2\pi^2 J_0\ell\right)^{3/2}}}. \quad (39)$$

Optimal protocols with $J = const$

Let $S(0) = S_0$ and $\dot{S}(0) = \dot{S}_0$. A process with $J = const$ is possible only for $J = 0$. Phase transition to the static case:

$$\mathcal{L}_J(\tau) = 0. \tag{40}$$

This is confirmed also by (28) at $J \to 0$.

Summary

No optimal finite-time quasi-static processes with fixed J on the classically unstable manifold $\mathcal{E}$ with respect to Weinhold's metric. Intrinsically non-equilibrium and non-reversible thermodynamics.

Entropy representation

Entropy in (M, J) space:

$$S = \frac{\pi\left(\sqrt{6\ell\left(5\nu^2+3\right)\left(6G\nu M^2\ell - J\left(\nu^2+3\right)\right)} + 12\sqrt{G}\nu^{3/2}M\ell\right)}{3\left(\nu^2+3\right)\sqrt{G\nu}}. \qquad (41)$$

Ruppeiner metric:

$$\hat{g} = \epsilon \begin{pmatrix} -\frac{2\sqrt{6}\pi J\sqrt{G\nu\left(5\nu^2+3\right)\ell^3}}{\left(6G\nu M^2\ell - J\left(\nu^2+3\right)\right)^{3/2}} & \frac{\sqrt{6}\pi M\sqrt{G\nu\left(5\nu^2+3\right)\ell^3}}{\left(6G\nu M^2\ell - J\left(\nu^2+3\right)\right)^{3/2}} \\ \frac{\sqrt{6}\pi M\sqrt{G\nu\left(5\nu^2+3\right)\ell^3}}{\left(6G\nu M^2\ell - J\left(\nu^2+3\right)\right)^{3/2}} & -\frac{\pi\left(\nu^2+3\right)\sqrt{5\nu^2\ell+3\ell}}{2\sqrt{6G\nu}\left(6G\nu M^2\ell - J\left(\nu^2+3\right)\right)^{3/2}} \end{pmatrix}. \qquad (42)$$

Thermodynamic curvature:

$$R_R = \frac{\left(\nu^2+3\right)\sqrt{3G\nu}}{\pi\epsilon\sqrt{2\ell\left(5\nu^2+3\right)\left(6G\nu M^2\ell - J\left(\nu^2+3\right)\right)}}. \qquad (43)$$

Extremality:

$$\frac{J}{M^2} = \frac{6G\ell\nu}{\nu^2+3}. \qquad (44)$$

Optimal processes with $M = const$

No optimal proceses with $J = const$!

Optimal processes with $M = const$ $(\epsilon = -1, J < J_0)$:

$$\mathcal{L}_M(J_0, J) = -\frac{2i\sqrt{\pi\epsilon}\,\sqrt[4]{2\ell(5\nu^2+3)}\,\sqrt[4]{6G\nu M^2\ell - J\left(\nu^2+3\right)}}{\sqrt[4]{3G\nu}\sqrt{\nu^2+3}}\Bigg|_{J_0}^{J}. \quad (45)$$

Geodesics at $M = const$ and $Y_0 = 6G\nu M^2\ell - J_0\left(\nu^2+3\right)$:

$$\ddot{J}(t) + \frac{3\left(\nu^2+3\right)\dot{J}(t)^2}{24G\nu M^2\ell - 4\left(\nu^2+3\right)J(t)} = 0, \quad J(0) = J_0, \quad \dot{J}(0) = \dot{J}_0, \quad (46)$$

$$J(t) = \dot{J}_0 t - \frac{Y_0 - 6G\nu M^2\ell}{\nu^2+3} - \frac{3\dot{J}_0^2\left(\nu^2+3\right)t^2}{8Y_0} + \frac{\dot{J}_0^3\left(\nu^2+3\right)^2 t^3}{16Y_0^2} - \frac{\dot{J}_0^4\left(\nu^2+3\right)^3 t^4}{256Y_0^3},$$

$$\mathcal{L}_M(\tau) = -\frac{i|\dot{J}_0|\,\sqrt[4]{\ell(5\nu^2+3)}\sqrt{\pi\epsilon\left(\nu^2+3\right)}}{\sqrt[4]{24G\nu Y_0^3}}\tau = \upsilon\tau, \quad \epsilon = -1. \quad (47)$$

$$\tau = \frac{4Y_0^{3/4}\left(\sqrt[4]{Y} - \sqrt[4]{Y_0}\right)}{\left(\nu^2+3\right)|\dot{J}_0|}, \quad Y|Y_0 = 6G\nu M^2\ell - \left(\nu^2+3\right)J|J_0. \quad (48)$$

Credits

Thank You!

- Partially supported by
  - SUMMIT № BG-RRP-2.004-0008-C01
  - The Sofia University Grant 80-10-150